MATH 320 NOTES, WEEK 1

Preliminary notation:

We write $x \in A$ to mean that an element x is in a set A. Examples:

(1) $1 \in \{-1, 0, 1, 3\}$
(2) $2 \notin \{-1, 0, 1, 3\}$,
(3) $2020 \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is the set of the natural numbers.
(4) $\pi \in \mathbb{R}$, where $\mathbb{R}$ is the set of the real numbers.

The empty set, $\emptyset$, is the set with no elements.

Given two sets A and B, we say that A is **a subset** of B, denoted by $A \subset B$ if every element of A also belongs to B.

Examples:

(1) $A \subset A$ and $\emptyset \subset A$ for any set A.
(2) $\{1, 2, 3\} \subset \mathbb{N}$;
(3) $\mathbb{N} \subset \mathbb{R}$.

Set notation: we will often define a set by specifying some property of its elements. For example let E be the set of all even natural numbers. We can write $E = \{2n \mid n \in \mathbb{N}\}$. Equivalently, we can write $E = \{k \mid k \in \mathbb{N}, 2 \text{ divides } k\}$.

Section 1.2 Vector Spaces

V is **a vector space** over a *filed* F, if there are operations

- (vector addition) $+$ from $V \times V$ to V, and
- (scalar multiplication) $\cdot$ from $F \times V$ to V,

such that the following axioms hold:

(1) For all x, y in V, $x + y = y + x$;
(2) For all x, y, z in V, $(x + y) + z = x + (y + z)$;
(3) There is a zero vector $\vec{0}$ in V, such that for all x in V, $x + \vec{0} = x$;
(4) For every x in V, there is a vector y, such that $x + y = \vec{0}$;
(5) For every x in V, $1x = x$;
(6) For every x in V and scalars a, b in F, $(ab)x = a(bx)$;
(7) For every x, y in V and every scalar a in F, $a(x + y) = ax + ay$;
(8) For every x in V and scalars a, b in F, $(a + b)x = ax + bx$.

The elements of the field F are often referred to as *scalars*.

Fields: A field has two commutative operations $+$ and $\cdot$, two identity elements 0 and 1, such that every element has a additive inverse and every nonzero element has a multiplicative inverse.

Examples of fields:

- The rational numbers $\mathbb{Q}$;
- The real numbers $\mathbb{R}$;
- The complex numbers $\mathbb{C}$;
- The field with characteristic 2, $F_2 = \{0, 1\}$.

The integers $\mathbb{Z}$ are not a field.

Next we go over some examples of vector spaces, starting with one of the simplest.

Definition 1. $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$, *i.e. this is the set of all pairs of real numbers.*

Lemma 2. $\mathbb{R}^2$ *is a field over* $\mathbb{R}$ *with the usual operations:*

- *vector addition:* $(a, b) + (c, d) = (a + b, c + d)$*;*
- *scalar multiplication:* $e \cdot (a, b) = (ea, eb)$*;*

We leave the proof as an exercise. Note that a, b, c, d, e above are all reals. Question:

(1) Is $\mathbb{R}^2$ a vector space over $\mathbb{C}$? That means we can multiply by complex scalars.
(2) Is $\mathbb{R}^2$ a vector space over $\mathbb{Q}$? That means we can multiply only by rational scalars.

Similarly, define:

(1) $\mathbb{R}^3$ is all triples of real numbers;
(2) $\mathbb{R}^n = \{(a_1, ..., a_n) \mid a_1 \in \mathbb{R}, ..., a_n \in \mathbb{R}\}$ is all n-tuples of reals;
(3) $\mathbb{Q}^n = \{(a_1, ..., a_n) \mid a_1 \in \mathbb{Q}, ..., a_n \in \mathbb{Q}\}$ is all n-tuples of rational numbers;
(4) $\mathbb{C}^n = \{(a_1, ..., a_n) \mid a_1 \in \mathbb{C}, ..., a_n \in \mathbb{C}\}$ is all n-tuples of complex numbers;

More generally, we have the following classic example of a vector space:

Definition 3. *Let* F *be a field and* n *a natural number.*

$$F^n = \{(a_1, ..., a_n) \mid a_1 \in F, ..., a_n \in F\}$$

i.e. all n*-tuples of elements in* F*. Then* F^n *is a vector space over* F *with the usual operations:*

- *vector addition:* $(a_1, ..., a_n) + (b_1, ..., b_n) = (a_1 + b_1, ..., a_n + b_n)$*;*
- *scalar multiplication:* $c \cdot (a_1, ..., a_n) = (ca_1, ..., ca_n)$*;*

Matrices

An m by n matrix is a two dimensional array of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

We will also denote $A = (a_{ij})$ for a matrix as above. If $A = (a_{ij})$ and $B = (b_{ij})$ are two m by n matrices, and c is a scalar, set

- $A + B = (a_{ij} + b_{ij})$
- $cA = (ca_{ij})$

Definition 4. *$M_{m,n}(F)$ is the set of all m by n matrices with entries in F. $M_{m,n}(F)$ is a vector space over F with the matrix addition and scalar operation defined above.*

Let us verify some of the axioms:

- Matrix addition is commutative and associative by definition.
- The zero vector is the zero matrix, i.e the matrix where each entry is 0.
- Given a matrix $A = (a_{ij})$, the negative inverse is $-A = (-a_{ij})$.
- One can check the associativity in multiplication, the distributive laws and that $1A = A$.

Polynomials Let F be a field. A polynomial with coefficients in F of degree n is of the form:

$$p(x) = a_n x^n + ... + a_1 x + a_0$$

where $a_0, ..., a_n \in F$ and $a_n \neq 0$. We can also write $p(x) = \Sigma_{i=0}^n a_i x^i$.

Given two polynomials $p(x) = a_n x^n + ... + a_1 x + a_0$ and $q(x) = b_n x^n + ... + b_1 x + b_0$, and a scalar c, define:

- $p(x) + q(x) = (a_n + b_n)x^n + ... + (a_1 + b_1)x + a_0 + b_0$,
- $cp(x) = ca_n x^n + ... + ca_1 x + ca_0$.

Examples of vector spaces:

(1) The set of all polynomial with coefficients in F is a vector space over F with the above operations. Denote it by $P(F)$.
(2) The set of all polynomials of degree $\leq n$ with coefficients in F is a vector space over F with the above operations. Denote it by $P_n(F)$.

Let us verify some of the axioms:

- Polynomial addition is commutative and associative by definition.
- The zero vector is the zero polynomial.
- Given $p(x) = a_n x^n + ... + a_1 x + a_0$, the negative inverse is $-a_n x^n + ... + (-a_1)x - a_0$.
- One can check the associativity in multiplication, the distributive laws and that $1p(x) = p(x)$.

Question: Is the set of all polynomials of a fixed degree a vector space? Why or why not?

Next we state our first theorem about vector spaces. Note that while the statement may seem obvious, it is important that we can prove it using *only* the axioms of vector spaces.

Theorem 5. *Suppose that x, y, z are vectors in a vector space V over F, such that $x + z = y + z$. Then $x = y$.*

Proof. Let w be the additive inverse of z i.e. the vector such that $z + w = \vec{0}$. Then,

$$x = x + \vec{0} = x + (z + w) = (x + z) + w = (y + z) + w = y + (z + w) = y + \vec{0} = y.$$

The first equality is by Axiom 3 (the zero vector existence), the second equality is by Axiom 4 (additive inverse), the third equality is by Axiom 2 (associativity of addition); then we use the assumption of the theorem, and then we use again Axioms 2, 4, 3 in that order. □

Corollary 6. *Let V be a vector space.*

(1) *The zero vector is unique.*
(2) *The additive inverse is unique.*

Lemma 7. *Let V be a vector space over F. Let $x \in V$ and $a \in F$.*

(1) $0x = \vec{0}$ *for every* $x \in V$;
(2) $(-a)x = -(ax) = a(-x)$;
(3) $a\vec{0} = \vec{0}$.

Proof. We give the proof of the first item, the rest is an exercise.

$0x + 0x = (0 + 0)x = 0x = 0x + \vec{0}$, so by Theorem 5, $0x = \vec{0}$.

□

Section 1.3 Subspaces

Let V be a vector space over a field F. Suppose that $W \subset V$. If W itself is a vector space over F, with the operations inherited from V, then we say that W is **a subspace** of V.

For example, the space of all polynomials of degree $\leq n$ with coefficients in F, $P_n(F)$, is a subspace of $P(F)$, the space of all polynomials. In another example, we will see that $W = \{(a, 0) \mid a \in \mathbb{R}\}$ is a subset of $\mathbb{R}^2$. (Visually, W can be identified the x-axis in the coordinate plane $\mathbb{R}^2$.)

In this section we will learn how to determine if a subset of a vector space is a subspace or not. Fortunately, it turns out that we do not have to check if every single axiom holds for W. Since we know the axioms hold for the bigger V, it is enough to check that the two operations (vector addition and scalar multiplication) are well defined when restricted to W. I.e. that W is closed under the operations. This is stated in following key theorem:

Theorem 8. *Suppose that V is a vector space over a field F and $W \subset V$. Then W is a subspace iff*

(1) $\vec{0} \in W$;
(2) *(closure under vector addition) If $x, y \in W$, then $x + y \in W$;*
(3) *(closure under scalar multiplication) If $x \in W$ and $c \in F$, then $cx \in W$.*

Example: Prove that $W = \{(0, b, 2b) \mid b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Proof. We check each of the three items in the above theorem.
(1). $\vec{0} = (0, 0, 0) \in W$ by definition (i.e. plug in $b = 0$).
(2). Suppose that $x, y \in W$. Say $x = (0, b, 2b)$ and $y = (0, c, 2c)$. Then

$$x + y = (0 + 0, b + c, 2b + 2c) = (0, b + c, 2(b + c)) \in W.$$

(3) Suppose $x = (0, b, 2b) \in W$ and $a \in F$. Then $ax = (0, ab, 2ab) \in W$. □

Definition 9. *Given an m by n matrix $A = (a_{ij}) \in M_{m,n}(F)$, its* **transpose**, *$A^t$ is the n be m matrix such that, denoting $A^t = (b_{ij})$, we have $b_{ij} = a_{ji}$. An n by n matrix A is* **symmetric** *if $A^t = A$. I.e. $A = (a_{ij})$ is symmetric iff for all $1 \le i, j \le n$, $a_{ij} = a_{ji}$*

Example 2: Prove that the set of symmetric matrices is a subspace of $M_{n,n}(F)$.

Proof. We check each of the three items in the above theorem.
(1). $O^t = O$, so the zero matrix is symmetric.
(2). Closure under matrix addition: suppose that A, B are two symmetric matrices. Say $A = (a_ij)$ and $B = (b_{ij})$. Denote $A + B = (c_{ij})$. Then $c_{ij} = a_{ij} + b_{ij} = a_{ji} + b_{ji} = c_{ji}$, so $A + B$ is also symmetric.
(3) Closure under scalar multiplication: suppose $A = (a_{ij})$ is symmetric, and $c \in F$. Then the (i, j)-th entry of cA is $ca_{ij} = ca_{ji}$, which equals the (j, i)-th entry of cA. So cA is also symmetric. □

Definition 10. *Let $A = (a_{ij}) \in M_{n,n}(F)$.*
(1) *A is* **a diagonal matrix** *if whenever $i \neq j$, $a_{ij} = 0$, i.e. every non-diagonal entry has to be zero.*
(2) *A is* **upper triangular** *if whenever $i > j$, $a_{ij} = 0$, i.e. every entry below the diagonal has to be zero.*
(3) *The* **trace** *of A is $tr(A) = a_{11} + a_{22} + ... + a_{nn}$.*

Now we can see some more examples of subspaces of $M_{n,n}(F)$:
(1) The set of upper triangular matrices.
(2) The set of diagonal matrices.
(3) $\{A \mid tr(A) = 0\}$.